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On the Dynamic Energy Release Rate  
in Elastic Crack Propagation

by

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the dynamic theory) of the well known relations

$$\begin{aligned} \epsilon(t_0) &= -\frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{C_{t_0 t}} \underline{s}(\underline{x}, t_0) \cdot \underline{u}(\underline{x}, t) d\underline{A}_x, \\ &= -\frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{t_0 t} \underline{s}(\underline{x}, t_0) \cdot \underline{u}(\underline{x} - \underline{l}(t), t_0) d\underline{A}_x, \end{aligned}$$

where  $\underline{s}$  is the surface traction,  $\underline{u}$  is the displacement,  $C_{t_0 t}$  is the portion of the crack generated in the time interval  $[t_0, t]$ , and  $\underline{l}(t) = \underline{z}_t - \underline{z}_{t_0}$  with  $\underline{z}_t$  at the position of the crack tip at time  $t$ .

To simplify our analysis, we avoid geometrical and notational complications by limiting our discussion to edge cracks in two-dimensional bodies. Also, our analysis is based on classical smoothness hypotheses and therefore in applying our results care must be taken to insure that the underling neighborhood of the crack tip is free of shock waves, etc.

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On the Dynamic Energy Release Rate in  
Elastic Crack Propagation

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1. Introduction

It is the purpose of this paper to give a unified treatment of the dynamic energy release rate,  $\mathcal{E}$ , for a sharp, straight crack in a hyperelastic body undergoing finite strain. As our main result we decompose  $\mathcal{E}$  into the usual quasi-static energy release rate plus a nonpositive dynamic contribution; thus for a dynamic solution the energy release rate computed using the classical quasi-static formula is larger than the actual dynamic energy release rate. We also present what are apparently the first proofs (within the dynamic theory) of the well known relations

$$\begin{aligned}\mathcal{E}(t_0) &= -\frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{C_{t_0 t}} \underline{s}(\underline{x}, t_0) \cdot \underline{u}(\underline{x}, t) dA_{\underline{x}}, \\ &= -\frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{C_{t_0 t}} \underline{s}(\underline{x}, t_0) \cdot \underline{u}(\underline{x} - \underline{l}(t), t_0) dA_{\underline{x}},\end{aligned}$$

where  $\underline{s}$  is the surface traction,  $\underline{u}$  is the displacement,  $C_{t_0 t}$  is the portion of the crack generated in the time interval  $[t_0, t]$ , and  $\underline{l}(t) = \underline{z}_t - \underline{z}_{t_0}$  with  $\underline{z}_t$  the position of the crack tip at time  $t$ .

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To simplify our analysis, we avoid geometrical and notational complications by limiting our discussion to edge cracks in two-dimensional bodies. Also, our analysis is based on classical smoothness hypotheses<sup>1</sup> and therefore in applying our results care must be taken to insure that the underlying neighborhood of the crack tip is free of shock waves, etc.

Notation. Light-face letters indicate scalars; bold-face lower case letters indicate vectors (in  $\mathbb{R}^2$ ); bold-face upper case letters indicate second-order tensors (linear transformations from  $\mathbb{R}^2$  into  $\mathbb{R}^2$ );  $\underline{A}^T$  is the transpose of  $\underline{A}$ ;  $\underline{A} \cdot \underline{B} = A_{ij}B_{ij}$ ;<sup>2</sup>  $\text{div } \underline{S}$  is the vector with components  $\partial S_{ij}/\partial x_j$ ;  $\nabla \underline{u}$  is the tensor with components  $\partial u_i/\partial x_j$ ;  $\nabla \nabla \underline{u}$  is the third-order tensor with components  $\partial^2 u_i/\partial x_j \partial x_k$ ; a superposed dot denotes differentiation with respect to time;  $L^p(\mathcal{R})$  is the class of all functions  $\varphi$  on  $\mathcal{R}$  with  $|\varphi|^p$  integrable on  $\mathcal{R}$ .

<sup>1</sup>Such hypotheses are tacit in most other studies of this type. An exception is Freund [1977].

<sup>2</sup>Here we use standard indicial notation and cartesian coordinates.

## 2. Basic equations.

To fix notation we consider first a two-dimensional regular body  $B$ , which we identify with the regular region of  $\mathbb{R}^2$  it occupies in a fixed reference configuration. We assume that the body is hyperelastic,<sup>1</sup> so that the displacement  $\underline{u}(\underline{x}, t)$ , the (Piola-Kirchhoff) stress  $\underline{s}(\underline{x}, t)$ , and the stored energy  $w(\underline{x}, t)$  obey the energy equation

$$\dot{w} = \underline{s} \cdot \nabla \dot{\underline{u}} \quad (2.1)$$

and the equation of motion

$$\operatorname{div} \underline{s} = \rho \ddot{\underline{u}} \quad (2.2)$$

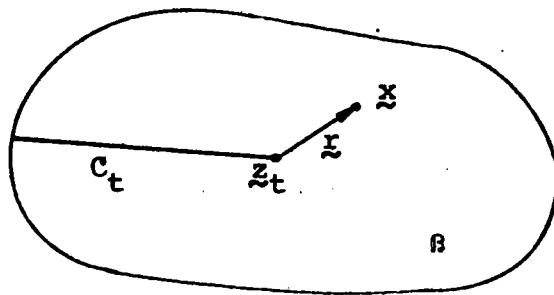
with  $\rho > 0$  the density in the reference configuration. We assume throughout that  $\rho$  is constant.

The above equations are appropriate to both the finite and infinitesimal theories of elasticity. In the infinitesimal theory  $\underline{s}$  is symmetric and  $w$  quadratic, but these restrictions are not relevant to most of what follows.

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<sup>1</sup>With the exception of Section 6, our analysis is valid for more general materials provided one uses (2.1) as the definition of  $w$ .

### 3. Mathematical preliminaries.



We consider a fixed time interval  $[0, T]$ . We assume that<sup>1</sup>  $B$  contains a straight edge crack  $C_t$ . The tip of the crack at time  $t$  is denoted by  $z_t$ ; we assume that  $z_t$  is  $C^2$  in  $t$  with velocity

$$\dot{z}(t) = \frac{d}{dt} z_t \neq 0 \quad (3.1)$$

and that

$$z_t \in B^o$$

for  $0 \leq t \leq T$ , where  $B^o$  is the interior of  $B$ .

The fields  $\varphi(x, t)$  of interest will be defined at each  $x$  in  $B \setminus C_t$  and each  $t \in [0, T]$ . A field of this type is a  $C^n$  fracture field ( $n \geq 0$  an integer) if:

- (i) the derivatives of  $\varphi$  of order  $\leq n$  exist away from the crack;
- (ii)  $\varphi$  and its derivatives of order  $\leq n$  are continuous away from the crack and, except at the tip, are continuous up to the crack from either side.

We write

$$\varphi \in L^p(B)$$

<sup>1</sup>It is important to note that  $B$  here need not be the entire body, but rather an arbitrarily small neighborhood of the tip (cf. the remark preceding Theorem 1).

if  $\varphi(\cdot, t) \in L^P(\mathcal{B})$  at each  $t \in [0, T]$ . If  $\varphi \in L^P(\mathcal{B})$ , then given any one-parameter family  $\mathcal{B}_\delta$  ( $\delta > 0$ ) of regular subregions of  $\mathcal{B}$  with  $\text{area}(\mathcal{B} \setminus \mathcal{B}_\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ ,

$$\int_{\mathcal{B}_\delta} |\varphi(\underline{x}, t)|^P da \rightarrow \int_{\mathcal{B}} |\varphi(\underline{x}, t)|^P da$$

as  $\delta \rightarrow 0$ ; when this limit is uniform in  $t \in [0, T]$ , we say that  $\varphi \in L^P(\mathcal{B})$  uniformly in time. Analogous interpretations apply to the assertions  $\varphi \in L^P(\partial\mathcal{B})$ ,  $\varphi \in L^P(\partial\mathcal{B})$  uniformly in time, etc.

Consider now  $\varphi(\underline{x}, t)$  as a function  $\varphi(\underline{z}_t + \underline{x}, t)$  of  $t$  and the position vector  $\underline{x}$  from the tip. We let  $\varphi'$  denote the derivative of this function with respect to  $t$  holding  $\underline{x}$  fixed; thus, by (3.1),

$$\varphi' = \dot{\varphi} + \nabla_{\underline{c}} \varphi \quad (3.2)$$

with

$$\nabla_{\underline{c}} \varphi = \nabla \varphi \cdot \underline{c}$$

the directional derivative of  $\varphi$  in the direction  $\underline{c}$ . For a vector field  $\underline{u}$ ,  $\underline{u}'$  is defined in the same manner, except that now

$$\nabla_{\underline{c}} \underline{u} = (\nabla \underline{u}) \underline{c}.$$

Since  $\mathcal{C}_t \subset \mathcal{C}_T$  for  $0 \leq t \leq T$ , each  $\mathcal{C}^n$  fracture field is of class  $\mathcal{C}^n$  on the cartesian product  $(\mathcal{B} \setminus \mathcal{C}_T) \times [0, T]$ . The next two lemmas give certain important identities for functions of this type.

Lemma 1. Let  $\varphi$  be a  $C^1$  scalar field on<sup>1</sup>  $(\mathcal{B} \setminus C_T) \times [0, T]$ .  
Assume that  $\varphi, \dot{\varphi} \in L^1(\mathcal{B})$  uniformly in time. Then

$$\int_{\mathcal{B}} \varphi \, da \quad (3.3)$$

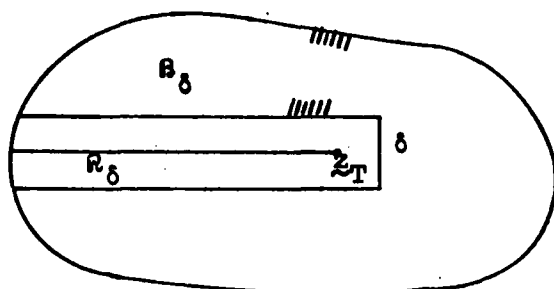
is a  $C^1$  function of time and

$$\frac{d}{dt} \int_{\mathcal{B}} \varphi \, da = \int_{\mathcal{B}} \dot{\varphi} \, da. \quad (3.4)$$

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<sup>1</sup>Of course,  $\varphi$  may have singularities on  $C_T$ .

Proof. Let  $R_\delta$  be the region shown whose boundary consists



of a portion of  $\partial B$ , two lines parallel to  $C_T$ , and a straight end face of length  $\delta$  perpendicular to these lines and such that  $C_T$  is contained in  $\overset{\circ}{R}_\delta$ . Let

$B_\delta = B \setminus R_\delta$ , so that  $\varphi$  is  $C^1$  on  $\overline{B}_\delta \times [0, T]$ . Then

$$\frac{d}{dt} \int_{B_\delta} \varphi \, da = \int_{B_\delta} \dot{\varphi} \, da. \quad (3.5)$$

Moreover, as  $\dot{\varphi} \in L^1(B)$  uniformly in time,

$$\int_{B_\delta} \dot{\varphi} \, da \rightarrow \int_B \dot{\varphi} \, da \quad (3.6)$$

as  $\delta \rightarrow 0$ , uniformly in time. Thus, since

$$\int_{B_\delta} \varphi \, da \rightarrow \int_B \varphi \, da, \quad (3.7)$$

(3.4) holds. Further, since the left side of (3.6) is continuous and the convergence uniform, the right side must also be continuous; hence (3.3) is  $C^1$  in time.  $\square$

Henceforth, in boundary integrals the letter  $n$  will always designate the outward unit normal.

Lemma 2. Let  $\varphi, \psi,$  and  $w$  be scalar fields on  $(B \setminus C_T) \times [0, T]$ . Assume that:

(i)  $\varphi$  and  $w$  are  $C^1$  and

$$\dot{\varphi} = \psi + \nabla_C w;$$

(ii)  $\varphi, \psi \in L^1(B)$  and  $w \in L^1(\partial B)$ , all uniformly in time.

Then

$$\int_B \varphi \, da$$

is  $C^1$  in time and

$$\frac{d}{dt} \int_B \varphi \, da = \int_B \psi \, da + \int_{\partial B} w \, \underline{c} \cdot \underline{n} \, d\mathcal{A}.$$

Proof. Let  $R_\delta$  and  $B_\delta$  be as in the previous proof. On the upper and lower horizontal portions of  $\partial R_\delta$ ,  $\underline{c} \cdot \underline{n} = 0$ , where  $\underline{n}$  is the outward unit normal to  $\partial B_\delta$ . Also, by the continuity of  $w$  away from the crack, the integral of  $w \underline{c} \cdot \underline{n}$  over the vertical right end face of  $R_\delta$  tends to zero as  $\delta \rightarrow 0$  uniformly in time. Thus (ii) and the divergence theorem imply that

$$\int_{\partial B} w \underline{c} \cdot \underline{n} \, d\mathcal{A} = \lim_{\delta \rightarrow 0} \int_{\partial B_\delta} w \underline{c} \cdot \underline{n} \, d\mathcal{A} = \lim_{\delta \rightarrow 0} \int_{B_\delta} \nabla_C w \, da;$$

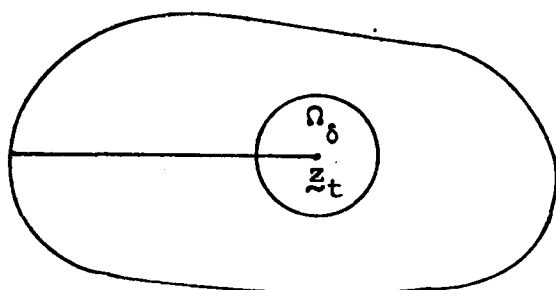
hence (i) and (ii) yield

$$\int_B \psi \, da + \int_{\partial B} w \underline{c} \cdot \underline{n} \, d\mathcal{A} = \lim_{\delta \rightarrow 0} \int_{B_\delta} (\psi + \nabla_C w) \, da = \lim_{\delta \rightarrow 0} \int_{B_\delta} \dot{\varphi} \, da,$$

and this limit is uniform in time. But (3.5) and (3.7) hold in the present circumstances. Hence

$$\frac{d}{dt} \int_{\mathcal{R}} \varphi \, da = \lim_{\delta \rightarrow 0} \int_{\mathcal{R}_\delta} \dot{\varphi} \, da = \int_{\mathcal{R}} \psi \, da + \int_{\partial \mathcal{R}} w_{\underline{c}} \cdot \underline{n} \, d\mathcal{A},$$

and, as in the proof of Lemma 1, the left side is continuous in time.  $\square$



In the next lemma, and in the sequel,  $\Omega_\delta = \Omega_\delta(t)$  is the disc of radius  $\delta$  centered at the crack tip,  $z_t$ .

Lemma 3. Let  $\underline{f}$  be a  $C^1$  fracture field with  $\operatorname{div} \underline{f} \in L^1(\mathcal{R})$ .  
Assume further that

$$\lim_{\delta \rightarrow 0} \int_{\partial \Omega_\delta} \underline{f} \cdot \underline{n} \, d\mathcal{A} = 0,$$

and that, for  $\underline{e}$  a unit vector normal to  $C_t$ ,  $\underline{f}(\underline{x}, t) \cdot \underline{e} \rightarrow 0$  as  $\underline{x}$  approaches  $C_t \setminus \{z_t\}$  from either side. Then

$$\int_{\partial \mathcal{R}} \underline{f} \cdot \underline{n} \, d\mathcal{A} = \int_{\mathcal{R}} \operatorname{div} \underline{f} \, da.$$

Proof. Simply apply the divergence theorem to the region  $\mathcal{R} \setminus \Omega_\delta$  and let  $\delta \rightarrow 0$ .  $\square$

#### 4. Basic assumptions. Energy release rate.

We begin by stating our assumptions concerning the fields  $\underline{u}(\underline{x}, t)$ ,  $\underline{s}(\underline{x}, t)$ , and  $w(\underline{x}, t)$ . Here and in what follows the field  $\underline{s}$  in a boundary integral will always denote the surface traction

$$\underline{s} = \underline{S} \underline{n}. \quad (4.1)$$

(A<sub>1</sub>)  $\underline{u}$  is a  $C^3$  fracture field;  $\underline{s}$  and  $w$  are  $C^1$  fracture fields;  $\underline{u}$ ,  $\underline{s}$ , and  $w$  obey (2.1) and (2.2) away from the crack.

(A<sub>2</sub>) For  $\underline{f} = \underline{u}, \underline{u}', \underline{u}'', \underline{u}'''$ :  $\underline{f}$  is bounded;  $\underline{s}, \nabla \underline{f} \in L^2(\mathcal{B})$  uniformly in time;  $w, w', \nabla \nabla \underline{f} \in L^1(\mathcal{B})$  uniformly in time.

(A<sub>3</sub>) The surface traction vanishes on the crack;<sup>1</sup> that is, if  $\underline{e}$  is a unit vector normal to  $C_t$ , then  $\underline{s}(\underline{x}, t) \underline{e} \rightarrow 0$  as  $\underline{x}$  approaches  $C_t \setminus \{z_t\}$  from either side.

(A<sub>4</sub>) Given any bounded vector field  $\underline{y}$  on  $\mathcal{B}$ ,

$$\lim_{\delta \rightarrow 0} \int_{\partial \Omega_\delta} \underline{s} \cdot \underline{y} \, d\mathcal{A} = 0.$$

Assumption (A<sub>4</sub>) asserts that, for bounded "velocity fields", the virtual power vanishes at the tip. By taking  $\underline{y}$  equal to a rigid velocity field one immediately concludes that (A<sub>4</sub>) rules out the possibility of a concentrated force or moment at the tip.

<sup>1</sup>Our results extend trivially to the case in which tractions are prescribed over  $C_0$ , the initial configuration of the crack: we simply replace integrals over  $\partial \mathcal{B}$  involving  $\underline{s}$  by corresponding integrals over  $\partial \mathcal{B} + C_0$ . Here an integral over  $C_0$  has the obvious meaning in terms of integrals over the "two faces" of  $C_0$  (cf. (6.4)).

Finally,  $(A_4)$  is implied by the somewhat more stringent assumption

$$\lim_{\delta \rightarrow 0} \int_{\partial \Omega_\delta} |\underline{s}| d\mathcal{A} = 0.$$

The function  $\mathcal{E}$  on  $[0, T]$  defined by

$$\mathcal{E} = \int_{\partial B} \underline{s} \cdot \dot{\underline{u}} d\mathcal{A} - \frac{d}{dt} \int_B (w+k) da \quad (4.2)$$

is called the dynamic energy release rate. Here

$$k = \frac{\rho}{2} \dot{\underline{u}}^2$$

is the kinetic energy per unit volume.

To see that  $\mathcal{E}$  is well defined, note first that, by  $(A_2)$  and the identity

$$\dot{\underline{u}} = \underline{u}' - \nabla_c \underline{u}, \quad (4.3)$$

$k$  satisfies

$$k, k' \in L^1(B) \text{ uniformly in time.}$$

Thus the existence of the second term<sup>1</sup> in (4.2) follows from the equation

$$(w+k)' = (w+k)' - \nabla_c (w+k)$$

and Lemma 2 with  $\varphi = -w = w+k$ ,  $\psi = (w+k)'$ .

---

<sup>1</sup>The first term is well defined.

Let  $\mathcal{R}$  be a regular subregion of  $\mathcal{B}$ . We say that  $\mathcal{R}$  surrounds the tip at time  $t$  if  $z_t \in \mathcal{R}$  and  $\partial\mathcal{R}$  intersects  $C_t$  at only one point.

Remark. Let  $\mathcal{R}$  surround the tip at time  $t_0$ . Then in a sufficiently small neighborhood of  $t = t_0$ , (2.1), (2.2), and the divergence theorem yield the energy equation

$$\int_{\partial(\mathcal{B} \setminus \mathcal{R})} \underline{s} \cdot \dot{\underline{u}} \, d\mathcal{A} = \frac{d}{dt} \int_{\mathcal{B} \setminus \mathcal{R}} (w+k) \, da,$$

and we can rewrite (4.2) in the form

$$\mathcal{E} = \int_{\partial\mathcal{R}} \underline{s} \cdot \dot{\underline{u}} \, d\mathcal{A} - \frac{d}{dt} \int_{\mathcal{R}} (w+k) \, da.$$

Thus  $\mathcal{E}$  is intrinsic to the crack tip; that is, the definition of  $\mathcal{E}$  is independent of the region  $\mathcal{R}$  surrounding the tip.

Our first result gives an alternative formula for  $\mathcal{E}$  in terms of a boundary integral over a region which shrinks to the crack tip,  $z_t$ . With this in mind we give the following definition. Let  $\Phi$  be a scalar-valued set function defined on the class of all regular subregions of  $\mathcal{B}$ . Let  $\Phi_0 \in \mathbb{R}$ . We write

$$\Phi_0 = \lim_{\mathcal{R} \rightarrow z_t} \Phi(\mathcal{R})$$

if given any  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$|\Phi_0 - \Phi(\mathcal{R})| < \varepsilon$$

for every region  $\mathcal{R}$  which surrounds the tip at time  $t$  and has area less than  $\delta$ .

Theorem 1. At each time  $t$ ,

$$\mathcal{E}(t) = \lim_{\mathcal{R} \rightarrow \mathcal{Z}_t} \int_{\partial \mathcal{R}} \left\{ \left( w + \frac{\rho}{2} |\nabla_{\mathcal{C}} \mathbf{u}|^2 \right) \underline{\mathbf{e}} - \underline{\mathbf{s}}^T \nabla_{\mathcal{C}} \mathbf{u} \right\} \cdot \underline{\mathbf{n}} \, d\mathcal{A} \quad (4.4)^1$$

Proof. By (2.2),

$$\operatorname{div}(\underline{\mathbf{s}}^T \underline{\mathbf{u}}') = \underline{\mathbf{s}} \cdot \nabla \underline{\mathbf{u}}' + \rho \ddot{\mathbf{u}} \cdot \underline{\mathbf{u}}', \quad (4.5)$$

and, since

$$\ddot{\mathbf{u}} = \underline{\mathbf{u}}'' - 2\nabla_{\mathcal{C}} \underline{\mathbf{u}}' + \nabla_{\mathcal{C}}^2 \underline{\mathbf{u}} - \nabla_{\mathcal{C}} \dot{\mathbf{u}}, \quad (4.6)$$

(A<sub>2</sub>) implies that the right side of (4.5) belongs to  $L^1(\mathcal{R})$ . Thus (A<sub>3</sub>), (A<sub>4</sub>), and Lemma 3 imply that

$$\int_{\partial \mathcal{R}} \underline{\mathbf{s}} \cdot \underline{\mathbf{u}}' \, d\mathcal{A} = \int_{\mathcal{R}} (\underline{\mathbf{s}} \cdot \nabla \underline{\mathbf{u}}' + \rho \ddot{\mathbf{u}} \cdot \underline{\mathbf{u}}') \, d\mathcal{A}. \quad (4.7)$$

Next, by (3.2), (4.6), and (A<sub>2</sub>),

$$\dot{\mathbf{w}} = \mathbf{w}' - \nabla_{\mathcal{C}} \mathbf{w}, \quad \dot{\mathbf{k}} = \mathbf{k} - \nabla_{\mathcal{C}} \left( \frac{\rho}{2} |\nabla_{\mathcal{C}} \mathbf{u}|^2 \right)$$

with

$$\mathbf{k} = \rho \ddot{\mathbf{u}} \cdot \underline{\mathbf{u}}' - \rho (\underline{\mathbf{u}}'' - 2\nabla_{\mathcal{C}} \underline{\mathbf{u}}' - \nabla_{\mathcal{C}} \dot{\mathbf{u}}) \cdot \nabla_{\mathcal{C}} \underline{\mathbf{u}} \in L^1(\mathcal{R}). \quad (4.8)$$

Thus Lemma 2 with

$$\varphi = \mathbf{w} + \mathbf{k}, \quad \psi = \mathbf{w}' + \mathbf{k}, \quad \omega = -\mathbf{w} - \frac{\rho}{2} |\nabla_{\mathcal{C}} \mathbf{u}|^2$$

<sup>1</sup>For a linear elastic solid a similar relation (containing  $-\dot{\mathbf{u}}$  in place of  $\nabla_{\mathcal{C}} \mathbf{u}$ ) appears, without proof, as Eqt. (13) of Atkinson and Eshelby [1968], and, with a sketch of a proof, as Eqt. (13) of Freund [1972].

yields

$$\frac{d}{dt} \int_{\mathcal{B}} (w+k) da = \int_{\mathcal{B}} (w'+k) da - \int_{\partial \mathcal{B}} \left( w + \frac{\rho}{2} |\nabla_{\mathcal{C}} u|^2 \right) \underline{s} \cdot \underline{n} d\sigma,$$

and this relation, (4.2), (4.3), and (4.7) imply

$$\mathcal{E} = \Phi(\mathcal{B}) - \int_{\mathcal{B}} (w' + \kappa - \underline{s} \cdot \nabla u' - \rho \ddot{u} \cdot u') da, \quad (4.9)$$

where  $\Phi(\mathcal{B})$  designates the integral in (4.4).

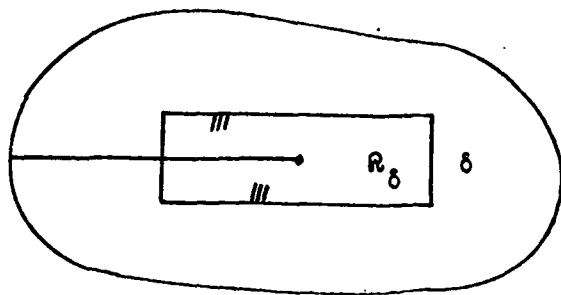
Now let  $\mathcal{R}$  surround the tip at time  $t$ , where  $t$  is fixed. Then the remark preceding Theorem 1 implies that  $\mathcal{B}$  in (4.9) can be replaced by  $\mathcal{R}$ :

$$\mathcal{E} = \Phi(\mathcal{R}) - \int_{\mathcal{R}} (w' + \kappa - \underline{s} \cdot \nabla u' - \rho \ddot{u} \cdot u') da.$$

Since the above integrand belongs to  $L^1(\mathcal{R})$ , (cf. (A<sub>2</sub>)) and

(4.8)), if we let the area of  $R$  approach zero, we arrive at the desired result (4.4).  $\square$

Remark. It is important to note that (by definition) the limit  $R \rightarrow z_t$  in (4.4) need not be confined to regions  $R$  whose diameters tend to zero. The chief requirement is that the area of  $R$  approach zero. Thus, if  $R_\delta$  is the one-parameter family



of regions shown in the figure, where each  $R_\delta$  is a rectangular region which surrounds the tip and has height  $\delta$ , width independent of  $\delta$ , and horizontal sides parallel to the crack, then

$$e(t) = \lim_{\delta \rightarrow 0} \int_{\partial R_\delta} \left\{ \left( w + \frac{\rho}{2} |\nabla u|^2 \right) \underline{s} - \underline{s}^T \nabla u \right\} \cdot \underline{n} \, d\mathcal{A};$$

or, since  $\underline{s} \cdot \underline{n}$  vanishes on the horizontal portions  $H_\delta$  of  $\partial R_\delta$ , while the integrals over the vertical portions tend to zero as  $\delta \rightarrow 0$ ,

$$e(t) = - \lim_{\delta \rightarrow 0} \int_{H_\delta} (\underline{s}^T \nabla u) \cdot \underline{n} \, d\mathcal{A},$$

$$= - \lim_{\delta \rightarrow 0} \underline{s} \cdot \int_{H_\delta} \nabla u^T \underline{s} \, d\mathcal{A}.$$

Henceforth,  $t_0$  is a fixed time in  $[0, T]$ . For convenience, we write

$$\left(\frac{d}{dt}\right)_{t_0} \quad (4.10)$$

for the derivative with respect to  $t$  at  $t = t_0$ . Also, given a fracture field  $\varphi$ , we write  $\varphi_t$  for the function on  $B \setminus C_t$  defined by

$$\varphi_t(x) = \varphi(x, t).$$

Theorem 2.

$$\mathcal{E}(t_0) = \left(\frac{d}{dt}\right)_{t_0} \int_B \int_{t_0}^t \{ (\underline{s}_t - \underline{s}_\lambda) \cdot \nabla \dot{\underline{u}}_\lambda + \rho (\ddot{\underline{u}}_t - \ddot{\underline{u}}_\lambda) \cdot \dot{\underline{u}}_\lambda \} d\lambda \, da. \quad (4.11)^1$$

Proof. Let

$$\mathfrak{F}(t) = \int_{t_0}^t \int_{\partial B} \underline{s} \cdot \dot{\underline{u}} \, d\mathcal{A} \, d\lambda - \int_B (w+k) \, da, \quad (4.12)$$

so that, by (4.2),

$$\mathcal{E} = \dot{\mathfrak{F}}. \quad (4.13)$$

The first term in (4.12) is equal to

$$\int_{t_0}^t \int_{\partial B} \underline{s}_t \cdot \dot{\underline{u}}_\lambda \, d\mathcal{A} \, d\lambda + \int_{t_0}^t \int_{\partial B} (\underline{s}_\lambda - \underline{s}_t) \cdot \dot{\underline{u}}_\lambda \, d\mathcal{A} \, d\lambda. \quad (4.14)$$

<sup>1</sup>Cf. Rice [1965], Eqt. (5). Rice considers infinitesimal strains and time-independent applied surface tractions.

Since

$$\frac{d}{dt} \int_{t_0}^t \int_{\partial B} (\underline{s}_\lambda - \underline{s}_t) \cdot \dot{\underline{u}}_\lambda \, d\mathcal{A} \, d\lambda = - \int_{t_0}^t \int_{\partial B} \dot{\underline{s}}_t \cdot \dot{\underline{u}}_\lambda \, d\mathcal{A} \, d\lambda,$$

(4.10) applied to the second term in (4.14) vanishes. Thus

$$\left(\frac{d}{dt}\right)_{t_0} \int_{t_0}^t \int_{\partial B} \underline{s} \cdot \dot{\underline{u}} \, d\mathcal{A} \, d\lambda = \left(\frac{d}{dt}\right)_{t_0} \int_{\partial B} \int_{t_0}^t \underline{s}_t \cdot \dot{\underline{u}}_\lambda \, d\lambda \, d\mathcal{A} = \left(\frac{d}{dt}\right)_{t_0} \int_{\partial B} \underline{s}_t \cdot (\underline{u}_t - \underline{u}_{t_0}) \, d\mathcal{A},$$

and, by (A<sub>2</sub>)-(A<sub>4</sub>), (2.2), (4.1), and the divergence theorem

(Lemma 3), this equals

$$\left(\frac{d}{dt}\right)_{t_0} \int_B \{ \underline{s}_t \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) + \rho \ddot{\underline{u}}_t \cdot (\underline{u}_t - \underline{u}_{t_0}) \} \, da; \quad (4.15)$$

hence (4.13) yields

$$\mathcal{E}(t_0) = \left(\frac{d}{dt}\right)_{t_0} \int_B \{ \underline{s}_t \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) + \rho \ddot{\underline{u}}_t \cdot (\underline{u}_t - \underline{u}_{t_0}) - (w+k) \} \, da. \quad (4.16)$$

Clearly,

$$\underline{s}_t \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) = \int_{t_0}^t \underline{s}_t \cdot \nabla \dot{\underline{u}}_\lambda \, d\lambda$$

for points  $\underline{x}$  not on  $C_t$ . Thus, since  $C_t$  is a set of (area) measure zero in  $B$ , both sides of this equation may be integrated over  $B$  to give

$$\int_B \underline{s}_t \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) \, da = \int_B \int_{t_0}^t \underline{s}_t \cdot \nabla \dot{\underline{u}}_\lambda \, d\lambda \, da.$$

Similarly,

$$\int_{\mathcal{R}} \rho \ddot{u}_t \cdot (u_t - u_{t_0}) da = \int_{\mathcal{R}} \int_{t_0}^t \rho \ddot{u}_t \cdot \dot{u}_\lambda d\lambda da,$$

and, using (2.1) and the definition of  $k$ ,

$$\frac{d}{dt} \int_{\mathcal{R}} (w+k) da = \frac{d}{dt} \int_{\mathcal{R}} \int_{t_0}^t (\dot{w} + \dot{k}) d\lambda da = \frac{d}{dt} \int_{\mathcal{R}} \int_{t_0}^t (\underline{s}_\lambda \cdot \nabla \dot{u}_\lambda + \rho \ddot{u}_\lambda \cdot \dot{u}_\lambda) d\lambda da.$$

The last three results and (4.16) yield (4.11).  $\square$

Remark.<sup>1</sup> It is clear from the remark preceding Theorem 1 that  $\mathcal{R}$  in (4.11) can be replaced by any region  $\mathcal{R}$  which surrounds the tip at time  $t_0$ .

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<sup>1</sup>There is much confusion in the literature concerning results of this type (cf. Gradin's [1979] correction of a misconception of Luxmoore and Morgan [1977] in the quasi-static theory).

### 5. Partition of $\mathcal{E}$ .

Theorem 2 allows us to write

$$\mathcal{E}(t_0) = \mathcal{U}(t_0) + \mathcal{K}(t_0)$$

with

$$\begin{aligned} \mathcal{U}(t_0) &= \left( \frac{d}{dt} \right)_{t_0} \int_B \int_{t_0}^t (\underline{S}_t - \underline{S}_\lambda) \cdot \nabla \dot{\underline{u}}_\lambda \, d\lambda \, da, \\ \mathcal{K}(t_0) &= \left( \frac{d}{dt} \right)_{t_0} \int_B \int_{t_0}^t \rho (\ddot{\underline{u}}_t - \ddot{\underline{u}}_\lambda) \cdot \dot{\underline{u}}_\lambda \, d\lambda \, da. \end{aligned} \tag{5.1}$$

$\mathcal{K}$  represents the contribution of inertial effects to the energy release rate, while  $\mathcal{U}$  gives the value the energy release rate would have were this dynamic contribution neglected.

For a linear elastic material, with positive semi-definite elasticity tensor, or, more generally, for a material which is stable in the sense of Drucker [1964],

$$\mathcal{U} \geq 0.$$

The next theorem, which is our main result, shows that, to the contrary,

$$\mathcal{K} \leq 0;$$

hence the effect of inertia is to reduce the energy release rate from the value obtained using the quasi-static formula  $(5.1)_1$ .

Theorem 3. Assume that  $u$  is continuous at  $t_0$  in the sense that

$$\lim_{t \rightarrow t_0} \sup_{x \in \beta} |u(x, t) - u(x, t_0)| = 0. \quad (5.2)$$

Then

$$\begin{aligned} K(t_0) &= \left( \frac{d}{dt} \right)_{t_0} \int_{\beta} \frac{\rho}{2} (\ddot{u}_t - \ddot{u}_{t_0}) \cdot (u_t - u_{t_0}) da, \\ &= \left( \frac{d}{dt} \right)_{t_0} \int_{\beta} \frac{\rho}{2} (u_t - u_{t_0}) \cdot \nabla_c^2(t) (u_t - u_{t_0}) da, \\ &= - \left( \frac{d}{dt} \right)_{t_0} \int_{\beta} \frac{\rho}{2} |\nabla_c(t) (u_t - u_{t_0})|^2 da; \end{aligned} \quad (5.3)$$

hence  $K(t_0) \leq 0$ .

Proof. Since

$$\int_{t_0}^t (\ddot{u}_t - \ddot{u}_{t_0}) \cdot \dot{u}_\lambda d\lambda = (\ddot{u}_t - \ddot{u}_{t_0}) \cdot (u_t - u_{t_0}),$$

we can write  $(5.1)_2$  as the right side of  $(5.3)_1$  plus  $\rho/2$  times

$$\left( \frac{d}{dt} \right)_{t_0} \int_{\beta} \varphi_1 da, \quad (5.4)$$

where

$$\varphi_1(x, t) = \int_{t_0}^t (\ddot{u}_t + \ddot{u}_{t_0} - 2\ddot{u}_\lambda) \cdot \dot{u}_\lambda d\lambda.$$

Thus, to establish  $(5.3)_1$  it suffices to show that  $(5.4)$  vanishes.

For convenience, let

$$u = u_t, \quad u_0 = u_{t_0}, \quad g = u - u_0, \quad \xi_0 = \xi(t_0).$$

A simple calculation then shows that

$$\varphi_1 = (\ddot{u} + \ddot{u}_0) \cdot g - (\dot{u}^2 - \dot{u}_0^2).$$

Next, in view of the identities (4.3), (4.6), and

$$\ddot{u} = u''' - 3\nabla_{\dot{u}} u'' - 3\nabla_{\dot{u}} u' - \nabla_{\dot{u}} u + 3\nabla_{\dot{u}}^2 u' + 3\nabla_c \nabla_{\dot{u}} u - \nabla_{\dot{u}}^3 u,$$

we have

$$(\varphi_1 - \varphi_2)' = \psi + \nabla_c w,$$

where

$$\varphi_2 = g \cdot (\nabla_c^2 - \nabla_c^2) u_0,$$

$$\begin{aligned} \psi = & [u''' - 3\nabla_{\dot{u}} u'' + 3\nabla_{\dot{u}}^2 u' - 3\nabla_{\dot{u}} u' + 3\nabla_c \nabla_{\dot{u}} u - \nabla_{\dot{u}}^3 u] \cdot g - (\ddot{u} - \ddot{u}_0) \cdot u' \\ & + [u'' - 2\nabla_{\dot{u}} u' - \nabla_{\dot{u}} u - (\ddot{u}_0 - 2\nabla_{\dot{u}_0} u'_0 - \nabla_{\dot{u}_0}^2 u_0)] \cdot \nabla_{\dot{u}} u \\ & + 2g \cdot \nabla_c \nabla_{\dot{u}} u_0 + u' \cdot (\nabla_c^2 - \nabla_{\dot{u}_0}^2) u_0, \end{aligned}$$

$$w = -g \cdot \nabla_{\dot{u}}^2 u + \nabla_{\dot{u}} u \cdot \nabla_c g.$$

It is easy to verify that  $\varphi = \varphi_1 - \varphi_2$ ,  $\psi$ , and  $w$  obey the hypotheses of Lemma 2; hence

$$\frac{d}{dt} \int_B (\varphi_1 - \varphi_2) da = \int_B \psi da + \int_{\partial B} w_{\dot{u}} \cdot \dot{u} d\mathcal{M},$$

and since

$$\psi(\underline{x}, t_0) = w(\underline{x}, t_0) = 0,$$

it follows that

$$\left(\frac{d}{dt}\right)_{t_0} \int_{\mathbb{R}} (\varphi_1 - \varphi_2) da = 0. \quad (5.5)$$

Next,

$$\int_{\mathbb{R}} \varphi_2 da = 0 \quad \text{at } t = t_0;$$

hence

$$\left(\frac{d}{dt}\right)_{t_0} \int_{\mathbb{R}} \varphi_2 da = \lim_{t \rightarrow t_0} \Psi(t_0, t),$$

where

$$\Psi(t_0, t) = \frac{1}{t - t_0} \int_{\mathbb{R}} \varphi_2(\underline{x}, t) da_{\underline{x}}.$$

Let  $\underline{g} = \underline{c}/|\underline{c}|$ . Then

$$\varphi_2 = \underline{g} \cdot (\underline{c}_0^2 - \underline{c}^2) \nabla_{\underline{e}_0}^2 u_0,$$

so that

$$|\Psi(t_0, t)| \leq \left(\sup_{\mathbb{R}} |\underline{u} - \underline{u}_0|\right) \left| \frac{\underline{c}_0^2 - \underline{c}^2}{t - t_0} \right| \int_{\mathbb{R}} |\nabla_{\underline{e}_0}^2 u_0| da,$$

and, by (5.2), this tends to zero as  $t \rightarrow t_0$ , since  $\nabla_{\underline{e}_0}^2 u_0 \in L^1(\mathbb{R})$

and

$$\left| \frac{\underline{c}_0^2 - \underline{c}^2}{t - t_0} \right| = \left| \frac{(\underline{c}_0 - \underline{c}) \cdot (\underline{c}_0 + \underline{c})}{t - t_0} \right| \rightarrow 2 |\underline{c}_0 \cdot \dot{\underline{c}}_0|.$$

Thus

$$\left(\frac{d}{dt}\right)_{t_0} \int_{\mathbb{R}} \varphi_2 da = 0 \quad (5.6)$$

and (5.5) implies that (5.4) is zero; hence (5.3)<sub>1</sub> holds.

Next, define

$$\varphi_3 = (\ddot{u} - \ddot{u}_0) \cdot g,$$

so that (5.3)<sub>1</sub> becomes

$$\kappa(t_0) = \frac{\rho}{2} \left(\frac{d}{dt}\right)_{t_0} \int_{\mathbb{R}} \varphi_3 da.$$

A simple calculation shows that

$$\varphi_3 = \gamma + g \cdot \nabla_c^2 g - \varphi_2,$$

$$\gamma = g \cdot \{ \ddot{u}'' - 2 \nabla_c \ddot{u}' - \nabla_c \ddot{u} - (\ddot{u}_0'' - 2 \nabla_c \ddot{u}_0' - \nabla_c \ddot{u}_0) \}, \quad (5.7)$$

and, by (A<sub>2</sub>),  $\gamma, \dot{\gamma} \in L^1(\mathbb{R})$  uniformly in time. Thus  $\gamma$  obeys the hypotheses of Lemma 1 and

$$\frac{d}{dt} \int_{\mathbb{R}} \gamma da = \int_{\mathbb{R}} \dot{\gamma} da.$$

But  $\dot{\gamma}(x, t_0) = 0$  for almost every  $x \in \mathbb{R}$ ; hence

$$\left(\frac{d}{dt}\right)_{t_0} \int_{\mathbb{R}} \gamma da = 0,$$

and, by (5.6) and (5.7)<sub>1</sub>, (5.3)<sub>2</sub> holds.

Next, let

$$\beta = \underline{g} \cdot \nabla_{\underline{C}} \underline{g} = \frac{1}{2} \nabla_{\underline{C}} (\underline{g}^2),$$

so that

$$\nabla_{\underline{C}} \beta = |\nabla_{\underline{C}} \underline{g}|^2 + \underline{g} \cdot \nabla_{\underline{C}}^2 \underline{g}. \quad (5.8)$$

Clearly,  $\nabla \beta \in L^1(\mathbb{R})$ . Consider the regions  $\mathbb{R}_\delta$  and  $\mathbb{B}_\delta$  introduced in the proof of Lemma 1. Using an argument similar to that given in the first paragraph of the proof of Lemma 2, we conclude that

$$\int_{\partial \mathbb{B}} \beta \underline{C} \cdot \underline{n} \, d\mathcal{A} = \lim_{\delta \rightarrow 0} \int_{\partial \mathbb{B}_\delta} \beta \underline{C} \cdot \underline{n} \, d\mathcal{A} = \lim_{\delta \rightarrow 0} \int_{\mathbb{B}_\delta} \nabla_{\underline{C}} \beta \, da = \int_{\mathbb{B}} \nabla_{\underline{C}} \beta \, da.$$

Since the left side of this relation is a  $C^1$  function of time, and since  $\beta$  and  $\dot{\beta}$  vanish at  $t = t_0$ ,

$$\left( \frac{d}{dt} \right)_{t_0} \int_{\mathbb{B}} \nabla_{\underline{C}} \beta \, da = \left( \frac{d}{dt} \right)_{t_0} \int_{\partial \mathbb{B}} \beta \underline{C} \cdot \underline{n} \, d\mathcal{A} = \int_{\partial \mathbb{B}} (\dot{\beta} \underline{C} \cdot \underline{n} + \beta \dot{\underline{C}} \cdot \underline{n}) \, d\mathcal{A} \Big|_{t=t_0} = 0;$$

thus we conclude from (5.8) that

$$\left( \frac{d}{dt} \right)_{t_0} \int_{\mathbb{B}} \underline{g} \cdot \nabla_{\underline{C}}^2 \underline{g} \, da = - \left( \frac{d}{dt} \right)_{t_0} \int_{\mathbb{B}} |\nabla_{\underline{C}} \underline{g}|^2 \, da,$$

and (5.3)<sub>3</sub> follows from (5.3)<sub>2</sub>.

Finally, since the integral in (5.3)<sub>3</sub> vanishes at  $t = 0$  and is  $\geq 0$  otherwise,  $\mathcal{K}(t_0) \leq 0$ .  $\square$

Remark. It should be emphasized that  $\mathcal{U}$  and  $\mathcal{K}$ , separately, do not generally represent rates of change of strain energy

$$W = \int_B w \, da$$

and kinetic energy<sup>1</sup>

$$K = \int_B k \, da.$$

Indeed, it is clear from (5.1) that

$$\begin{aligned} u(t_0) &= -\dot{W}(t_0) + \left(\frac{d}{dt}\right)_{t_0} \int_B \underline{s}_t \cdot (\underline{v}_{t_0} - \underline{v}_{t_0}) \, da, \\ K(t_0) &= -\dot{K}(t_0) + \left(\frac{d}{dt}\right)_{t_0} \int_B \rho \underline{u}_t \cdot (\underline{u}_t - \underline{u}_{t_0}) \, da. \end{aligned} \quad (5.9)$$

When the power supplied by the environment vanishes, i.e., when

$$\int_{\partial B} \underline{s} \cdot \underline{\dot{u}} \, dA = 0, \quad (5.10)$$

the last two terms in (5.9) sum to zero (cf. (4.15)), but are generally not individually zero, so that while

$$u + K = -\dot{W} - \dot{K},$$

in general

$$u \neq -\dot{W}, \quad K \neq -\dot{K}.$$

This is in contrast to the quasi-static theory for which  $u = -\dot{W}$  when (5.10) holds.

<sup>1</sup>Cf. the theoretical studies of Hahn, Hoagland, Rosenfield, and Sejnoha [1974], Freund [1977], and Popelar and Gehlen [1979], where actual curves are given for strain energy and kinetic energy as functions of crack length.

Remark. The definitions (5.1) are intrinsic to the tip; that is, (5.1) are invariant when  $\mathcal{B}$  is replaced by an arbitrary region  $\mathcal{R}$  surrounding the tip. Indeed, if we replace each integral over  $\mathcal{B}$  in (5.1) by an integral over  $\mathcal{R}$  plus an integral over  $\mathcal{B} \setminus \mathcal{R}$ , and interchange this integral over  $\mathcal{B} \setminus \mathcal{R}$  with the time derivative at  $t = t_0$ , we find that the terms involving  $\mathcal{B} \setminus \mathcal{R}$  vanish.

In view of this remark and the remark preceding Theorem 1,  $\mathcal{E}$ ,  $\mathcal{U}$ , and  $\mathcal{K}$  are possibly more closely related to the dynamical behavior of the crack than  $\dot{\mathcal{W}}$  and  $\dot{\mathcal{K}}$ , since  $\dot{\mathcal{W}}$  and  $\dot{\mathcal{K}}$  are generally not intrinsic to the tip.

# 6. The energy release rate for a linear elastic material.

We now restrict our attention to the linear theory for which

$$\underline{s} = \underline{c} \nabla \underline{u}, \quad w = \frac{1}{2} \nabla \underline{u} \cdot \underline{c} \nabla \underline{u} \quad (6.1)$$

with  $\underline{c}$ , the elasticity tensor, a symmetric linear transformation (at each point of  $\mathcal{B}$ ). Of course, symmetry is the requirement that

$$\underline{A} \cdot \underline{CB} = \underline{B} \cdot \underline{CA}$$

for all second-order tensors  $\underline{A}$  and  $\underline{B}$ .

## Theorem 4.

$$\begin{aligned} \dot{u}(t_0) &= \frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{\mathcal{B}} (\underline{s}_t - \underline{s}_{t_0}) \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) da, \\ &= \frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{\mathcal{B}} (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) \cdot \underline{c} (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) da, \quad (6.2)^1 \\ \dot{e}(t_0) &= \frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{\mathcal{B}} \{ (\underline{s}_t - \underline{s}_{t_0}) \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) + \rho (\dot{\underline{u}}_t - \dot{\underline{u}}_{t_0}) \cdot (\underline{u}_t - \underline{u}_{t_0}) \} da \end{aligned}$$

Proof. As in the first few steps of the proof of Theorem 3, we can decompose  $\dot{u}(t_0)$  as follows:

$$\dot{u}(t_0) = \frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \left\{ \int_{\mathcal{B}} (\underline{s}_t - \underline{s}_{t_0}) \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) da + \int_{\mathcal{B}} \int_{t_0}^t (\underline{s}_t + \underline{s}_{t_0} - 2\underline{s}_\lambda) \cdot \nabla \dot{\underline{u}}_\lambda d\lambda da \right\}.$$

Since  $\underline{c}$  is symmetric,

$$\underline{s}_t \cdot \nabla \underline{u}_{t_0} = \underline{s}_{t_0} \cdot \nabla \underline{u}_t,$$

<sup>1</sup>For (6.2)<sub>1</sub> cf. Bueckner [1958], Eq. (15); Rice [1965], Eq. (19).

and thus, by (2.1) and (6.1),

$$\int_{t_0}^t (\underline{s}_t + \underline{s}_{t_0} - 2\underline{s}_\lambda) \cdot \nabla \dot{\underline{u}}_\lambda \, d\lambda = (\underline{s}_t + \underline{s}_{t_0}) \cdot (\nabla \underline{u}_t - \nabla \underline{u}_{t_0}) - 2(w_t - w_{t_0}) = 0,$$

which yields (6.2)<sub>1,2</sub>. The result (6.2)<sub>3</sub> is a direct consequence of (5.3)<sub>1</sub> and (6.2)<sub>1</sub>.  $\square$

For  $t \geq t_0$  let

$$C_{t_0 t} = C_t \setminus C_{t_0}$$

denote the portion of the crack between  $\underline{z}_{t_0}$  and  $\underline{z}_t$ . In the statement of the next theorem we write

$$\int_{C_{t_0 t}} \underline{s}_{t_0} \cdot \underline{u}_t \, d\mathcal{A} \quad (6.3)$$

for the integral of  $\underline{s}_{t_0} \cdot \underline{u}_t$  over the "two faces" of  $C_{t_0 t}$ ; that is, writing

$$\underline{u}^+(\underline{x}, t) = \lim_{\delta \rightarrow 0^+} \underline{u}(\underline{x} + \delta \underline{n}^+, t),$$

$$\underline{s}^+(\underline{x}, t) = \lim_{\delta \rightarrow 0^+} \underline{s}(\underline{x} + \delta \underline{n}^+, t) \underline{n}^+,$$

for all  $\underline{x} \in C_T$  ( $\underline{x} \neq \underline{z}_t$ ), where  $\underline{n}^\pm = \pm \underline{e}$  with  $\underline{e}$  a unit vector perpendicular to the crack, then (6.3) is defined to be

$$\int_{C_{t_0 t}} \underline{s}_{t_0}^+ \cdot \underline{u}_t^+ \, d\mathcal{A} + \int_{C_{t_0 t}} \underline{s}_{t_0}^- \cdot \underline{u}_t^- \, d\mathcal{A}. \quad (6.4)$$

Also, in what follows the notation (4.10) designates the right-hand derivative at  $t_0$ .

Theorem 5. Assume that  $\tilde{s}_{t_0}^+ \in L^1(C_T)$ . Then

$$\begin{aligned} \mathcal{E}(t_0) &= -\frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{C_{t_0 t}} \tilde{s}_{t_0} \cdot \tilde{u}_t \, d\mathcal{A}, \\ &= -\frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{C_{t_0 t}} \tilde{s}(\tilde{x}, t_0) \cdot \tilde{u}(\tilde{x} - \tilde{l}(t), t_0) \, d\mathcal{A}_{\tilde{x}}, \end{aligned} \quad (6.5)^1$$

where  $\tilde{l}(t) = \tilde{z}_t - \tilde{z}_{t_0}$ .

Proof. By (6.2)<sub>3</sub>,

$$\mathcal{E}(t_0) = \frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int_{\mathcal{B}} \gamma \, da, \quad (6.6)$$

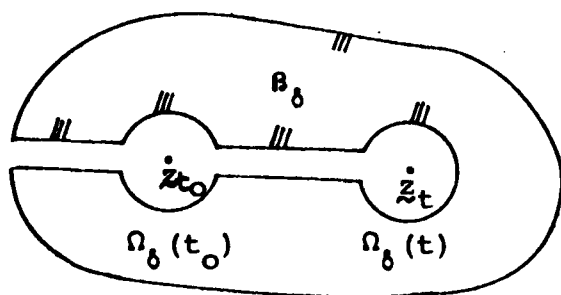
where

$$\gamma = (\tilde{s} - \tilde{s}_0) \cdot \nabla \tilde{g} + \rho(\tilde{u} - \tilde{u}_0) \cdot \tilde{g}. \quad (6.7)$$

(Recall that  $\tilde{g} = \tilde{u} - \tilde{u}_0$ .) Consider the region  $\mathcal{B}_\delta = \mathcal{B} \setminus \{\Omega_\delta(t_0) \cup \Omega_\delta(t)\}$  and fix  $t \in (t_0, T]$ . By (2.2),

$$\gamma = \operatorname{div}\{(\tilde{s} - \tilde{s}_0)^T \tilde{g}\}.$$

We therefore conclude from the divergence theorem and (A<sub>3</sub>) that



<sup>1</sup>For the quasi-static theory (6.5)<sub>1</sub> was derived by Bueckner [1958] (see also Rice [1965]), while (6.5)<sub>2</sub> is essentially contained in the work of Irwin [1957, 1958]. For the dynamic theory a heuristic argument in support of (6.5)<sub>2</sub> was given by Erdogan [1968]. The observation that (6.5) require separate proof in the dynamic theory, and that such a proof is probably non-trivial, is due to Freund [1972], who notes that (6.5) yield "the correct result for all problems for which a solution is known".

$$\int_{B_\delta} \gamma \, da = \int_{\partial B} (\underline{s} - \underline{s}_0) \cdot \underline{g} \, d\mathcal{A} - \int_{\partial \Omega_\delta(t_0)} (\underline{s} - \underline{s}_0) \cdot \underline{g} \, d\mathcal{A} - \int_{\partial \Omega_\delta(t)} (\underline{s} - \underline{s}_0) \cdot \underline{g} \, d\mathcal{A} + \int_{B_\delta \cap C_{t_0 t}} (\underline{s} - \underline{s}_0) \cdot \underline{g} \, d\mathcal{A}.$$

where the last integral has a meaning analogous to (6.4). Clearly,

$$\int_{\partial \Omega_\delta(t_0)} \underline{s} \cdot \underline{g} \, d\mathcal{A} \rightarrow 0$$

as  $\delta \rightarrow 0$ , because both  $\underline{s}$  and  $\underline{g}$  are bounded near  $\underline{z}_{t_0}$ . Similarly,

$$\int_{\partial \Omega_\delta(t)} \underline{s}_0 \cdot \underline{g} \, d\mathcal{A} \rightarrow 0.$$

On the other hand, since  $\underline{g}$  is bounded,  $(A_4)$  implies that

$$\int_{\partial \Omega_\delta(t_0)} \underline{s}_0 \cdot \underline{g} \, d\mathcal{A} \rightarrow 0, \quad \int_{\partial \Omega_\delta(t)} \underline{s} \cdot \underline{g} \, d\mathcal{A} \rightarrow 0,$$

and, by  $(A_3)$ ,

$$\int_{B_\delta \cap C_{t_0 t}} \underline{s} \cdot \underline{g} \, d\mathcal{A} = 0.$$

Further, since  $\underline{s}_0$  and  $\underline{u}_0$  are continuous across  $B_\delta \cap C_{t_0 t}$ ,

$$\int_{B_\delta \cap C_{t_0 t}} \underline{s}_0 \cdot \underline{u}_0 \, d\mathcal{A} = 0,$$

and, since  $\underline{s}_0^+ \in L^1(C_T)$  and  $\underline{u}$  is bounded,

$$\int_{B_\delta \cap C_{t_0 t}} \underline{s}_0 \cdot \underline{u} \, d\mathcal{A} \rightarrow \int_{C_{t_0 t}} \underline{s}_0 \cdot \underline{u} \, d\mathcal{A}.$$

Thus

$$\int_B \gamma \, da = \lim_{\delta \rightarrow 0} \int_{B_\delta} \gamma \, da = \int_{\partial B} (\underline{s} - \underline{s}_0) \cdot (\underline{u} - \underline{u}_0) \, d\mathcal{A} - \int_{C_{t_0 t}} \underline{s}_0 \cdot \underline{u} \, d\mathcal{A},$$

where we have used the fact that, by (6.7) and  $(A_2)$ ,  $\gamma \in L^1(B)$ .

This relation also holds at  $t = t_0$ , since (by definition) the integral over  $C_{t_0 t}$  is zero. Thus, since

$$\left(\frac{d}{dt}\right)_{t_0} \int_{\partial B} (\underline{s} - \underline{s}_0) \cdot (\underline{u} - \underline{u}_0) \, d\mathcal{A} = \int_{\partial B} \{ (\underline{s} - \underline{s}_0) \cdot \dot{\underline{u}} + \dot{\underline{s}} \cdot (\underline{u} - \underline{u}_0) \} \, d\mathcal{A} \Big|_{t=t_0} = 0,$$

$$\left(\frac{d}{dt}\right)_{t_0} \int_B \gamma \, da = - \left(\frac{d}{dt}\right)_{t_0} \int_{C_{t_0 t}} \underline{s}_0 \cdot \underline{u} \, d\mathcal{A},$$

this relation and (6.6) imply  $(6.5)_1$ .

To establish the equivalence of  $(6.5)_1$  and  $(6.5)_2$ , it clearly suffices to prove that

$$\lim_{t \rightarrow t_0^+} \int_{C_{t_0 t}} \underline{s}(\underline{x}, t_0) \cdot \left\{ \frac{\underline{u}(\underline{x}, t) - \underline{u}(\underline{x} - \underline{\ell}(t), t_0)}{t - t_0} \right\} d\mathcal{A}_{\underline{x}} = 0. \quad (6.8)$$

Writing  $\underline{x} = \underline{z}_t + \underline{r}$ , we see that

$$\underline{u}(\underline{x}, t) = \underline{u}(\underline{z}_t + \underline{r}, t),$$

$$\underline{u}(\underline{x} - \underline{\ell}(t), t_0) = \underline{u}(\underline{z}_{t_0} + \underline{r}, t_0).$$

Thus (cf. the paragraph containing (3.2)) the quantity in { }  
in (6.8) is bounded in absolute value by

$$\sup_{\mathbb{R} \times [0, T]} |\underline{u}'|,$$

which is finite by  $(A_2)$ . Thus, since  $\underline{g}_0^+ \in L^1(C_T)$ , (6.8) follows.  $\square$

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